Memory effects in recurrent and extreme events

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A dynamical approach to recurrent and extreme events is developed focusing on the role of correlations and memory in the structure of the probability distributions and their low-order moments. The procedure is illustrated on homogeneous first and second order Markov chains, non-Markovian and nonhomogeneous processes and deterministic dynamical systems. Substantial differences with classical statistical theory as applied to independent identically distributed random variables are identified.

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I. INTRODUCTION

Recurrent and extreme events are of great importance in a variety of fields. They signal the re-emergence of a particular configuration attained by a system at a certain time in the past or the occurrence of potentially catastrophic excursions of a relevant variable from its long term average and are, on these grounds, key elements to be accounted for when dealing with the issue of prediction of natural, technological or social systems [1,2].

There is a widespread feeling that owing to their scarcity, recurrent, and extreme events associated with a certain variable X monitored at regular times $t_n = n\Delta t$ are essentially independent, entailing that the set of the successive values X_n constitutes a set of independent identically distributed random variables (i.i.d.r.v.'s). There exists a powerful statistical theory devoted to this case which has met with an immense success, from hydrology to civil engineering to insurance and finance [3].

Recently, the present authors and co-workers have shown that recurrent and extreme events in deterministic dynamical systems [4] or in bimodal systems [5] display properties not accounted fully by the classical statistical theory. The objective of the present paper is to elaborate further on this theme by analyzing the role of memory in the probabilistic properties of recurrences and of extreme event related phenomena, such as successive exceedences. In addition to being one of the main signatures of the entire class of deterministic processes memory is also present in a variety of stochastic processes as well, a familiar example of which is provided by moving averages.

The types of situation we shall be concerned with are depicted in Figs. 1 and 2. The empty circles denote the available states, whereas the numbers next to them indicate the times of visit. We divide the full set of M states into two subsets containing K (subset C) and $M-K$ (subset $\bar{C}$) states, respectively, and convene that subsets C and $\bar{C}$ constitute two "cells" associated, in particular, with "subthreshold" (cell C) and "superthreshold" (cell $\bar{C}$) values of a representative variable. In this setting the paths in Figs. 1(a) and 1(b) correspond to recurrence in cell C and $\bar{C}$, respectively, whereas the path in Fig. 2 corresponds to an event of double exceedence of the threshold separating C and $\bar{C}$. To allow for the possibility of switching to a continuous time description ($\Delta t \to 0$) Smoluchowski's definition of recurrence and exceedence will be adopted [6]. Accordingly, the events in Figs. 1(a) and 2 will be conditioned by the probability of being in cell C at time 0 and in $\bar{C}$ at time 1 and likewise for Fig. 1(b).

We start in Sec. II with a brief survey of results based on the assumption of i.i.d.r.v.'s. In Sec. III we derive expressions for path probabilities and their moments in the case of homogeneous first- and second-order Markov processes, extending those of the classical theory. Non-Markovian and nonhomogeneous processes are considered in Sec. IV at dif-

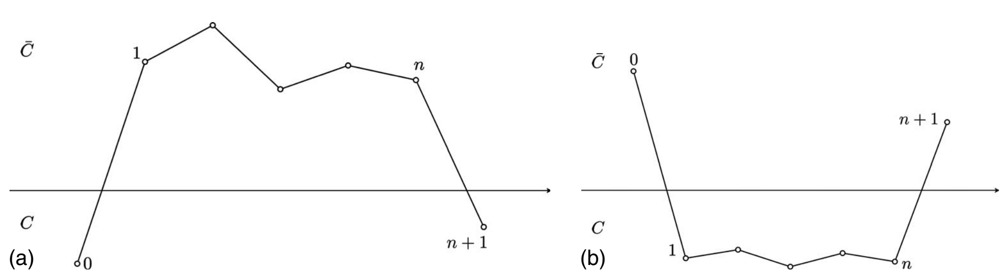

FIG. 1. Schematic representation of the phenomenon of recurrence within cells C and $\bar{C}$. The empty circles denote the instantaneous position of the trajectory whereas the numbers next to them the times spent outside the reference cell, starting initially from cell C (a) and $\bar{C}$ (b), respectively.

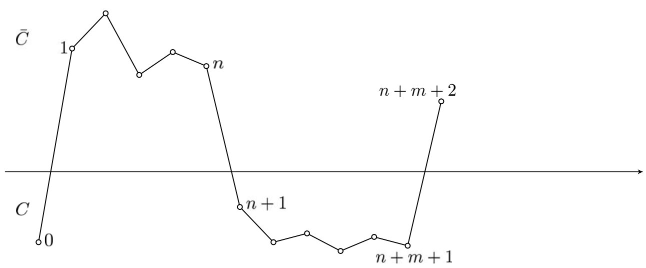

FIG. 2. Schematic representation of the phenomenon of double exceedence of a threshold starting initially at cell C.

ferent levels: first, the process induced when lumping the states of a Markov process; and second, processes in which the conditional probabilities of successive steps depend on the system's entire previous history. The connection with deterministic systems is discussed in Sec. V and the main conclusions are summarized in Sec. VI.

II. INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

As a reference and as a comparison point with the results derived subsequently for correlated processes we summarize in this section the probabilistic properties of the events depicted in Figs. 1 and 2 in the limit of i.i.d.r.v.'s. The main signature of statistical independence in this evaluation will be that the conditional probabilities for the successive steps will be equal to the absolute probabilities of the final states,

$$w(\bar{C}|C) = P_{\bar{C}}, \quad w(C|\bar{C}) = P_C. \tag{1}$$

In typical situations of interest the monitored variable X is a stochastic variable with continuous realizations. Let $\rho(x)$ be its probability density and

$$F_C = \int_C dx \rho(x), \quad F_{\bar{C}} = 1 - F_C \tag{2}$$

the associated cumulative distributions to be in cells C and $\bar{C}$, respectively. One has, then, $P_C = F_C$, $P_{\bar{C}} = 1 - F_C$ and the path and conditional probabilities (in the Smolukowski perspective, see Introduction) of the recurrent and exceedence events of Figs. 1 and 2 are as follows:

(i) Recurrence in C,

$$P_C^{\text{rec}}(n) = \text{Prob}(X_0 \in C, X_1 \in \bar{C}, \cdots X_n \in \bar{C}, X_{n+1} \in C)$$
$$= F_C(1 - F_C)^n F_C,$$

$$W_C^{\text{rec}}(n) = \frac{P_C^{\text{rec}}(n)}{F_C(1 - F_C)} = (1 - F_C)^{n-1} F_C. \tag{3}$$

(ii) Recurrence in $\bar{C}$,

$$P_{\bar{C}}^{\text{rec}}(m) = \text{Prob}(X_0 \in \bar{C}, X_1 \in C, \cdots X_m \in C, X_{m+1} \in \bar{C})$$
$$= (1 - F_C) F_C^m (1 - F_C),$$

$$W_{\bar{C}}^{\text{rec}}(m) = \frac{P_{\bar{C}}^{\text{rec}}(m)}{F_C(1 - F_C)} = F_C^{m-1}(1 - F_C). \tag{4}$$

(iii) Double exceedence of threshold from C to $\bar{C}$,

$$P^{\text{exc}}(n,m) = \text{Prob}(X_0 \in C, X_1 \in \bar{C}, \cdots X_n \in \bar{C}, X_{n+1} \in C, \cdots X_{n+m+1} \in C, X_{n+m+2} \in \bar{C})$$
$$= F_C(1 - F_C)^n F_C^m (1 - F_C),$$

$$W^{\text{exc}}(n,m) = \frac{P^{\text{exc}}(n,m)}{F_C(1 - F_C)} = (1 - F_C)^n F_C^n = W_C^{\text{rec}}(n) W_{\bar{C}}^{\text{rec}}(m). \tag{5}$$

All distributions (3)–(5) are, clearly, normalized when summed over n, over m and over both n and m, respectively. The corresponding means and variances can be evaluated straightforwardly, yielding

$$\tau_C^{\text{rec}} = \langle n \rangle = \frac{1}{F_C},$$

$$\tau_{\bar{C}}^{\text{rec}} = \langle m \rangle = \frac{1}{1 - F_C},$$

$$\tau^{\text{exc}} = \langle n + m \rangle = \tau_C^{\text{rec}} + \tau_{\bar{C}}^{\text{rec}} = \frac{1}{F_C} + \frac{1}{1 - F_C},$$

$$V_C^{\text{rec}} = \langle (\delta n)^2 \rangle = \frac{1 - F_C}{F_C^2},$$

$$V_{\bar{C}}^{\text{rec}} = \langle (\delta m)^2 \rangle = \frac{F_C}{(1 - F_C)^2}, \tag{6}$$

$$V^{\text{exc}} = \langle (\delta(n+m))^2 \rangle = \frac{3F_C^2 - 3F_C + 1}{F_C^2(1 - F_C)^2} = \langle (\delta n)^2 \rangle + \langle (\delta m)^2 \rangle. \tag{7}$$

Notice the additivity property in the last Eqs. (6) and (7), a direct consequence of the factorization property in Eq. (5).

III. HOMOGENEOUS MARKOV PROCESSES

As a first step toward assessing the role of correlations in recurrence and exceedence phenomena we consider in this section the case where the transitions between C and $\bar{C}$ constitute first and second order Markov processes, successively. The simplest implementation of this setting is provided by a discrete chain involving two states. A more intricate situation is that of a multistate chain in which the states are lumped into two groups, C and $\bar{C}$.

A. First-order Markov chain

We limit ourselves to irreducible chains consisting of ergodic states. The process is fully defined by the knowledge of the invariant probabilities P_C, $P_{\bar{C}}$ and of the one-step conditional probabilities, assumed to depend only on the time step associated with the transition (homogeneity property in time)

$$w(\bar{C}|C) = w_{\bar{C}C}, \quad w(C|\bar{C}) = w_{C\bar{C}} \tag{8a}$$

with

$$w(C|C) = w_{CC} = 1 - w_{\bar{C}C}, \qquad w\bar{C}|\bar{C}) = w_{\bar{C}\bar{C}} = 1 - w_{C\bar{C}}. \tag{8b}$$

The conditional probabilities (again, in the Smolukowski sense) for the events in Figs. 1 and 2 extending those in Eqs. (3) and (5) are now given by

$$W_C^{\text{rec}}(n) = w_{\bar{C}\bar{C}}^{n-1} w_{C\bar{C}} = (1 - w_{C\bar{C}})^{n-1} w_{C\bar{C}}, \tag{9}$$

$$W_{\bar{C}}^{\text{rec}}(m) = w_{CC}^{m-1} w_{\bar{C}C} = (1 - w_{\bar{C}C})^{m-1} w_{\bar{C}C}, \tag{10}$$

$$\begin{aligned} W^{\text{exc}}(n,m) &= (1 - w_{C\bar{C}})^{n-1} w_{C\bar{C}} (1 - w_{\bar{C}C})^{m-1} w_{\bar{C}C} \\ &= W_C^{\text{rec}}(n) W_{\bar{C}}^{\text{rec}}(m). \end{aligned} \tag{11}$$

Notice that the factorization property in Eq. (5) extends to the present case as well, even though (one-step) correlations are accounted for. The corresponding means and variances are on the other hand given by

$$\tau_C^{\text{rec}} = \langle n \rangle = \frac{1}{w_{C\bar{C}}},$$

$$\tau_{\bar{C}}^{\text{rec}} = \langle m \rangle = \frac{1}{w_{\bar{C}C}},$$

$$\tau^{\text{exc}} = \langle n + m \rangle = \frac{1}{w_{C\bar{C}}} + \frac{1}{w_{\bar{C}C}},$$

$$V_C^{\text{rec}} = \langle (\delta n)^2 \rangle = \frac{1 - w_{C\bar{C}}}{w_{C\bar{C}}^2},$$

$$V_{\bar{C}}^{\text{rec}} = \langle (\delta m)^2 \rangle = \frac{1 - w_{\bar{C}C}}{w_{\bar{C}C}^2}, \tag{12}$$

$$V^{\text{exc}} = \langle (\delta(n+m))^2 \rangle = \langle (\delta n)^2 \rangle + \langle (\delta m)^2 \rangle. \tag{13}$$

While the additivity property noticed in Sec. II still holds [last equations in Eqs. (12) and (13)] the expressions for the recurrence times now bear the signature of correlations, as they are not reducible to the absolute probabilities of being in cells C or $\bar{C}$ or, alternatively, to the measures of these cells.

Since the relevant variable in the perspective of double exceedences is the sum $n+m$, it is of interest to deduce from Eq. (11) the reduced probability distribution

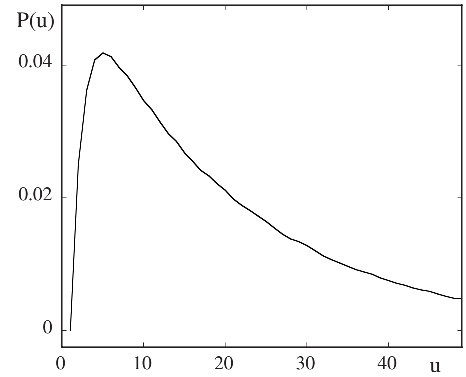

FIG. 3. Dependence of the probability P of the time u between two successive exceedences in the case of a two-state homogeneous Markov process as a function of u as obtained numerically after averaging over 10^6 realizations. Parameter values are $w_{CC}=0.5$ and $w_{\bar{C}\bar{C}}=0.95$.

$$P^{\text{exc}}(u) = \sum_{n,m=1}^{\infty} W^{\text{exc}}(n,m)\, \delta_{n+m,u}^{\text{kr}}. \tag{14}$$

Performing the Kronecker delta and using expression (11) we obtain

$$P(u) = \sum_{n=1}^{u-1} w_{\bar{C}\bar{C}}^{n-1} w_{CC}^{u-n-1} w_{C\bar{C}} w_{\bar{C}C}$$

or finally

$$P(u) = \frac{w_{C\bar{C}} w_{\bar{C}C}}{w_{\bar{C}\bar{C}} - w_{CC}} (w_{\bar{C}\bar{C}}^{u-1} - w_{CC}^{u-1}), \tag{15}$$

which is properly normalized. Interestingly, this distribution possesses a maximum at a u value given by

$$u_{\max} = \left(\ln \frac{w_{\bar{C}\bar{C}}}{w_{CC}} \right)^{-1} \ln \frac{w_{\bar{C}\bar{C}} \ln w_{CC}}{w_{CC} \ln w_{\bar{C}\bar{C}}}. \tag{16}$$

Figure 3 depicts the result of evaluation of $P(u)$ by a direct stochastic simulation of the process, in full agreement with the above analytical expressions.

B. Second-order Markov chain

Extending the memory by one additional step entails that for a full description of the evolution one needs, besides the invariant probabilities P_C and $P_{\bar{C}}$, the set of eight conditional probabilities,

$$w(C|A,B), \quad w(\bar{C}|A,B) \quad (A,B = C,\bar{C}) \tag{17a}$$

with

$$w(C|A,B) + w(\bar{C}|A,B) = 1. \tag{17b}$$

As in the previous subsection, time-homogeneous processes are again assumed.

An explicit model satisfying these properties, proposed by Raftery [7] consists in expressing w as a weighted sum of transitions to the present state from *either* of the two states 1 and 2 time units behind,

$$w(C|A,B) = \lambda q_{CA} + (1-\lambda) q_{CB},$$

$$w(\bar{C}|A,B) = \lambda q_{\bar{C}A} + (1-\lambda) q_{\bar{C}B}, \quad (18)$$

where $0 \le \lambda \le 1$ and q_{CA} etc. are the elements of a stochastic matrix.

To express the conditional probabilities for the events in Figs. 1 and 2 one needs now to distinguish between the case where both n and m are larger than equal to 2 and the case where at least one of them is equal to 1.

Recurrence in cell C

$$W_C^{\rm rec}(n=1) = w(C|\bar{C},C) = \lambda q_{C\bar{C}} + (1-\lambda) q_{CC}, \quad (19a)$$

$$W_C^{\rm rec}(n \ge 2) = w(\bar{C}|\bar{C},C) w^{n-2}(\bar{C}|\bar{C},\bar{C}) w(C|\bar{C},\bar{C}) = (\lambda q_{\bar{C}\bar{C}} + (1-\lambda) q_{\bar{C}C}) q_{\bar{C}\bar{C}}^{n-2} q_{C\bar{C}}, \quad (19b)$$

and likewise for cell $\bar{C}$.

Double exceedence

$$W^{\rm exc}(n=m=1) = w(C|\bar{C},C) w(\bar{C}|C,\bar{C}) = [\lambda q_{C\bar{C}} + (1-\lambda) q_{CC}][\lambda q_{\bar{C}C} + (1-\lambda) q_{\bar{C}\bar{C}}], \quad (20a)$$

$$W^{\rm exc}(n \ge 2, m=1) = w(\bar{C}|\bar{C},C) w^{n-2}(\bar{C}|\bar{C},\bar{C}) w(C|\bar{C},\bar{C}) w(\bar{C}|C,\bar{C}) = [\lambda q_{\bar{C}C} + (1-\lambda) q_{\bar{C}\bar{C}}][\lambda q_{\bar{C}\bar{C}} + (1-\lambda) q_{\bar{C}C}] q_{\bar{C}\bar{C}}^{n-2} q_{C\bar{C}}, \quad (20b)$$

$$W^{\rm exc}(n=1, m \ge 2) = w(C|\bar{C},C) w(C|C,\bar{C}) w^{m-2}(C|C,C) w(\bar{C}|C,C) = [\lambda q_{C\bar{C}} + (1-\lambda) q_{CC}][\lambda q_{CC} + (1-\lambda) q_{C\bar{C}}] q_{CC}^{m-2} q_{\bar{C}C}, \quad (20c)$$

$$W^{\rm exc}(n \ge 2, m \ge 2) = w(\bar{C}|\bar{C},C) w^{n-2}(\bar{C}|\bar{C},\bar{C}) w(C|\bar{C},\bar{C}) \times w(C|C,\bar{C}) w^{m-2}(C|C,C) w(\bar{C}|C,C) = [\lambda q_{\bar{C}\bar{C}} + (1-\lambda) q_{\bar{C}C}][\lambda q_{CC} + (1-\lambda) q_{C\bar{C}}] \times q_{\bar{C}\bar{C}}^{n-2} q_{C\bar{C}} q_{CC}^{m-2} q_{\bar{C}C}. \quad (20d)$$

On inspecting Eqs. (19) and (20) one sees that the factorization of $W^{\rm exc}$ into $W_C^{\rm rec}$ and $W_{\bar{C}}^{\rm rec}$ (and hence the additivity of the corresponding averages) holds once again, since the extension of memory affects both types of distribution in similar ways. There is, however, a signature of the extended memory in the expression of the mean recurrence and exceedence times. As an example the mean recurrence time in cell C reads

$$\tau_C^{\rm rec} = \frac{w(C|\bar{C},\bar{C}) + w(\bar{C}|\bar{C},C)}{w(C|\bar{C},\bar{C})} \quad (21)$$

reducing, as expected, to the first expression (12) in the limit of one-step Markov process for which the state two time units behind plays no more a role.

IV. NON-MARKOVIAN, NONHOMOGENEOUS PROCESSES

We consider, successively, a case of non-Markovian process generated by lumping of the states of a finite Markov chain into two subsets C and $\bar{C}$, a genuine non-Markovian process in which the transition between states C and $\bar{C}$ depends explicitly on past history and a process satisfying the Chapman-Kolmogorov equation where the property of time homogeneity breaks down.

A. Lumping states in a finite Markov chain

Consider once again the general setting of Figs. 1 and 2. Suppose that the transition between states is governed by a finite Markov chain and let s_i and f_j be two states such that $s_i \in C$ and $f_j \in \bar{C}$. By definition, the conditional probability for performing in a single step a transition from s_i to $\bar{C}$ is

$$w(\bar{C}|s_i) = \sum_{j \in \bar{C}} w(f_j|s_i) \quad (22)$$

and likewise for a transition from f_j to C. The converse of this process, namely, a transition from $\bar{C}$ to s_i (and likewise from C to f_j) obeys, on the other hand, to more involved rules unless all individual conditional probabilities satisfy the lumpability property,

$$w(s_i|f_j) = w(s_i|\bar{C}) \quad (\text{independent of } j),$$

$$w(f_j|s_i) = w(f_j|C) \quad (\text{independent of } i). \quad (23)$$

In this case lumping preserves the Markov property of the original chain [8] and the analysis of Sec. III A applies straightforwardly.

We now want to determine the consequences (if any) of a non-Markovian partitioning of the original states into two coarse-grained states C and $\bar{C}$, in the probabilistic properties of the recurrence and exceedence events considered in the present work. Let P_C, $P_{\bar{C}}$ be the probabilities of being in C and $\bar{C}$. By definition

$$P_C = \sum_{i \in C} P_i,$$

$$P_{\bar{C}} = \sum_{j \in \bar{C}} P_j, \quad (24)$$

where $\{P_i\}$, $\{P_j\}$ are the probabilities of the states of the original process. We next introduce the quantities

$$\tilde{W}_{\bar{C}C} = \sum_{i \in C,\ j \in \bar{C}} \tilde{P}_i w_{ji}, \quad (25a)$$

$$\tilde{W}_{C\bar{C}} = \sum_{j\in\bar{C},\ i\in C} \tilde{P}_j w_{ij},$$

with

$$\tilde{P}_i = \frac{P_i}{\sum_{i'\in C} P'_i}, \quad \tilde{P}_j = \frac{P_j}{\sum_{j'\in\bar{C}} P'_j}. \tag{25b}$$

If the lumpability conditions [Eqs. (23)] were satisfied, the quantities in Eq. (25a) would stand for the transition probabilities between states C and $\bar{C}$ of a *bona fide* two-state Markov chain. The question we ask is, whether in absence of lumpability they still determine the recurrence and exceedence properties. Surprisingly, the answer to this question is in the affirmative at least as far mean values [Eqs. (12)] are concerned. The reason for this is in fact that when dealing with recurrence or exceedence events in a stationary stochastic process one essentially probes (after transients have elapsed) properties depending on the invariant probability and the one-step transition probabilities. These quantities satisfy the relations

$$P_i = \sum_{j\in C\cup\bar{C}} w_{ij}P_j \quad i \in C \cup \bar{C}, \tag{26}$$

which entail in turn the following property involving the quantities defined in Eqs. (24) and (25):

$$\frac{P_C}{P_{\bar{C}}} = \frac{\tilde{W}_{\bar{C}C}}{\tilde{W}_{C\bar{C}}}. \tag{27}$$

This is nothing but the expression satisfied by the invariant probabilities of a two-state Markov chain with transition probabilities given by $w_{\bar{C}C}$ and $w_{C\bar{C}}$, and the statement is thereby proved. Alternatively, we here deal with a property of weak lumpability [8] valid as long as the starting probability vector is the invariant measure of the chain. This conclusion is fully corroborated by the results of a direct simulation of the stochastic process.

It should be pointed out that results as above break down when time-dependent properties are concerned. In this case, projecting the evolution equation

$$\mathbf{P}(n+1) = W \cdot \mathbf{P}(n)$$

with

$$\mathbf{P} = (P_1 \cdots P_M)^T, \quad W = \{w_{ij}\} \tag{28}$$

into the coarse-grained states [Eq. (24)] will no longer yield a closed equation involving solely $P_C(n)$ and $P_{\bar{C}}(n)$ for all times but, rather, an inhomogeneous equation involving a "complement" not belonging to the coarse-grained subspace. As an example consider a three-state Markov chain and let $C=(1,2)$ and $\bar{C}=(3)$. We define the matrices

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{29}$$

Applying U on both sides of Eq. (28) and using the property that UV is the unit matrix and VU a block-diagonal matrix involving exclusively stochastic submatrices one obtains an evolution equation of the form

$$\begin{pmatrix} P_C(n+1) \\ P_{\bar{C}}(n+1) \end{pmatrix} = \hat{W} \begin{pmatrix} P_C(n) \\ P_{\bar{C}}(n) \end{pmatrix} + \begin{pmatrix} q(n) \\ -q(n) \end{pmatrix}, \tag{30}$$

where $\hat{W}=UWV$ and $q(n)=\frac{w_{32}-w_{31}}{2}(P_1-P_2)$. This equation displays clearly the correction to be brought to the coarse-grained evolution governed by the matrix $\hat{W}$ when the lumpability condition $w_{32}=w_{31}$ [cf. Eq. (23)] is not satisfied. In the steady state $P_C(n+1)=P_C(n)$ and likewise for $P_{\bar{C}}$, and one recovers relations (24), (25a), (25b), (26), and (27).

B. Two-state non-Markovian process: The Polya model

Non-Markovian processes arise in a variety of problems from physics to finance, often in connection with generalized random walk type processes [9,10]. In this subsection we will be interested in the Polya model, a minimal two-state system in which extensive analytic results complemented with stochastic simulations can be obtained.

The model is defined as follows [11]. A box contains initially b black and r red balls. In the first time step a ball is drawn at random, is replaced and, moreover, c balls of the color drawn are added. A new random drawing from the box (which now contains $b+r+c$ balls) is made in the second time step and the procedure is repeated for as long as desired. We are interested in the recurrence and exceedence properties of the variable X_n taking two distinct values according to whether the nth drawing results in a black (state $C\equiv b$) or a red (state $\bar{C}\equiv r$) ball. Clearly, we are here dealing with a process where, contrary to those considered so far in this work, the transition probabilities are updated such that the drawing of either state increases its probability at the next drawing. This provides a rough model for such phenomena as the propagation of a contagion.

We are now in the position to write out the analytic expressions for the probabilities of the events depicted in Figs. 1 and 2.

(i) Recurrence in $C\equiv b$,

$$W_b^{\rm rec} = \frac{\text{path probability of event of having 2 black and } n \text{ red drawings}}{\text{probability of 1 black and 1 red drawing}}$$

or

$$W_b^{\rm rec} = \frac{(b+c)\Pi_{j=1}^{n-1}(r+jc)}{\Pi_{k=2}^{n+1}(b+r+kc)} = \frac{\left(\frac{b}{c}+1\right)\Gamma\left(\frac{r}{c}+n\right)\Gamma\left(\frac{b+r}{c}+2\right)}{\Gamma\left(\frac{r}{c}+1\right)\Gamma\left(\frac{b+r}{c}+2+n\right)}. \tag{31}$$

(ii) Recurrence in $\bar{C}\equiv r$,

$$W_r^{\rm rec} = \frac{\text{path probability of event of having } m \text{ black and 2 red drawings}}{\text{probability of 1 red and 1 black drawing}}$$

or

$$W_r^{\rm rec} = \frac{(r+c)\Pi_{j=1}^{m-1}(b+jc)}{\Pi_{k=2}^{m+1}(b+r+kc)} = \frac{\left(\frac{r}{c}+1\right)\Gamma\left(\frac{b+r}{c}+2\right)\Gamma\left(\frac{b}{c}+m\right)}{\Gamma\left(\frac{b}{c}+1\right)\Gamma\left(\frac{b+r}{c}+2+m\right)}. \tag{32}$$

(iii) Second passage probability from b to r (the analog of double exceedence),

$$W^{\rm exc}(n,m) = \frac{\text{path probability of event of having } m+1 \text{ black and } n+1 \text{ red drawings}}{\text{probability of 1 black and 1 red drawing}}$$

or

$$W^{\rm exc}(n,m) = \frac{\Pi_{j=1}^{n}(r+jc)\Pi_{k=1}^{m}(b+kc)}{\Pi_{\ell=2}^{n+m+1}(b+r+\ell c)} = \frac{\Gamma\left(\frac{b+r}{c}+2\right)\Gamma\left(\frac{b}{c}+1+m\right)\Gamma\left(\frac{r}{c}+1+n\right)}{\Gamma\left(\frac{r}{c}+1\right)\Gamma\left(\frac{b}{c}+1\right)\Gamma\left(\frac{b+r}{c}+2+n+m\right)} \tag{33}$$

All three distributions (31)–(33) are normalized to unity when summed over n, m and both n and m, respectively. They fall off as inverse powers of their arguments in the limit of large n and m, differing in this respect from the exponential form of their counterparts considered in Secs. II and III. We also notice that the factorization property of $W^{\rm exc}$ into $W_b^{\rm rec}$ and $W_r^{\rm rec}$ is no longer satisfied—a direct consequence of the updating of the successive transition probabilities and hence of the non-Markovian character of the process. Nevertheless, on computing the mean values one obtains the analytic result

$$\tau_b^{\rm rec} = \frac{b+r+c}{b},$$

$$\tau_r^{\rm rec} = \frac{b+r+c}{r},$$

$$\tau^{\rm exc} = \frac{(b+r)(b+r+c)}{br} = \tau_b^{\rm rec} + \tau_r^{\rm rec}. \tag{34}$$

This relative insensitivity of the mean to the non-Markovian character of the process reflects probably the fact that averaging only probes the global properties of the underlying probability distribution. On the other hand the computation of the covariance of m and n leads to a nonvanishing expression, reflecting the presence of correlations

$$\langle \delta m \, \delta n\rangle = -\frac{(b+r+c)c}{br}. \tag{35}$$

For $c=0$ this expression vanishes as expected, since in this limit the Polya process reduces to a Bernoulli process.

In a similar vein the variances of recurrence and exceedence times can be evaluated analytically, yielding

$$V_C^{\rm rec} = \frac{(b+c)(r+c)(b+r+c)}{(b-c)b^2},$$

$$V_{\bar{C}}^{\rm rec} = \frac{(b+c)(r+c)(b+r+c)}{(r-c)r^2},$$

$$V^{\rm exc} = V_C^{\rm rec} + V_{\bar{C}}^{\rm rec} - 2\frac{c(b+r+c)}{br}. \tag{36}$$

Additivity of the variances thus fails, contrary to the i.i.d.r.v. and Markov cases. The decrease of $V^{\rm exc}$ compared to the sum of variances of recurrences in C and $\bar{C}$ reflects the fact that computing moments associated with the quantity $n+m$ requires a contracted form of the full distribution $W^{\rm exc}(n,m)$, similar to the distribution $P(u)$ introduced in Eq. (14). This is further confirmed by the computation of the entropy of the distribution $P(u)$, which turns out to be less than the sum of entropies of $W_C^{\rm rec}$ and $W_{\bar{C}}^{\rm rec}$. Figure 4 depicts the dependence of $V^{\rm exc}$ on r as obtained from a direct stochastic simulation of the process, in full agreement with the analytical result in Eq. (36).

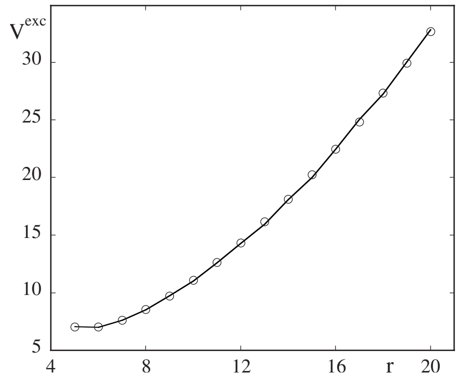

FIG. 4. Dependence of V^{exc} on r as obtained from a direct numerical simulation of the Polya process after averaging over 10^6 realizations (full line). Empty circles stand for the analytic result [Eq. (36)].

The nonindependence of successive exceedences in the Polya model has also been established using the method of Contingency Tables combined with a χ^2 test [12]. Let N be the total number of events recorded, $K_{n,m}$ the number of occurrences of the particular combination (n,m) and ν_n, ν_m the observed frequencies of recurrent events in C and $\bar{C}$. We compare the quantity

$$d^2 = \sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(K_{mn}-N\nu_n\nu_m)^2}{N\nu_n\nu_m} \tag{37}$$

with the value $\chi^2_{1-\alpha}(N^2-1)$, where $1-\alpha$ is a prescribed level of acceptance of the "null hypothesis" of the events being independent, and $\chi^2_{1-\alpha}$ is determined from the chi-square distribution. The null hypothesis will be accepted if and only if $d^2<\chi^2_{1-\alpha}(N^2-1)$. Choosing $\alpha=0.95$, $r=b=5$ one finds that the null hypothesis is to be unambiguously and definitely rejected for $c=1$ and retained (as expected) in the limit $c=0$ in which the Polya process reduces to a Bernoulli process.

C. Nonhomogeneous processes

We close this section with the analysis of double exceedences for processes in which the conditional probabilities depend not only on the time interval between initial and final states but also on time at which the initial state has occurred. This is the case, for instance, of systems subjected to time-dependent control parameters in the form of ramps (as it happens when switching a device), external fields, etc. It will be assumed that the process still satisfies the Chapman-Kolmogorov equation [11]. and reduces to two states each of which is associated to the subthreshold (state 1) and superthreshold (state 2) value of the relevant variable.

The specific model we adopt assigns the following form to the four conditional probabilities involved,

$$w_{11}=w(1,n+1|1,n)=\alpha_n, \quad w_{21}=1-\alpha_n, \tag{38a}$$

$$w_{22}=w(2,n+1|2,n)=\beta, \quad w_{12}=1-\beta. \tag{38b}$$

The path probabilities replacing expressions (31)–(33) become (cf. Figs. 1 and 2),

$$W_1^{\text{rec}}(n)=\beta^{n-1}(1-\beta), \tag{39a}$$

$$W_2^{\text{rec}}(m)=\alpha_1\cdots\alpha_{m-1}(1-\alpha_m), \tag{39b}$$

$$W^{\text{exc}}(n,m)=\beta^{n-1}(1-\beta)\alpha_{n+1}\cdots\alpha_{n+m}(1-\alpha_{n+m+1}), \tag{39c}$$

and are all properly normalized provided that β and α_n for any n remain strictly smaller than unity. Clearly, expression (39c) is not reducible to a product of Eqs. (39a) and (39b) as it carries the memory of the order in which exceedences took place. Multiplying Eqs. (39) by n,m and $(n+m)$ successively and summing over all n and m from 1 to ∞ we obtain

$$\tau_1^{\text{rec}}=\frac{1}{1-\beta}, \tag{40a}$$

$$\tau_2^{\text{rec}}=1+\sum_{n=1}^{\infty}\prod_{j=1}^{n}\alpha_j, \tag{40b}$$

$$\tau^{\text{exc}}=\frac{1}{1-\beta}+\sum_{n=1}^{\infty}\left\{1+\sum_{m=1}^{\infty}\prod_{j=n+1}^{n+m+1}\alpha_j\right\}\beta^{n-1}(1-\beta). \tag{40c}$$

On inspecting Eqs. (40a)–(40c) one is led to anticipate [keeping also in mind the nonfactorizability of $W^{\text{exc}}(n,m)$] that nonadditivity, $\tau^{\text{exc}}\neq\tau_1^{\text{rec}}+\tau_2^{\text{rec}}$ should hold for a typical form of α_n. This is confirmed by explicit calculations using the model

$$\alpha_n=\frac{q_1+q_2 n}{1+n} \quad 0<q_1,\ q_2<1. \tag{41}$$

Equation (40b) yields then

$$\tau_2^{\text{rec}}=\frac{(1-q_2)^{-q_1/q_2}-1}{q_1}, \tag{42}$$

whereas Eq. (40c) gives rise to a weighted sum of hypergeometric functions whose numerical evaluation for specific values of q_1, q_2, and β yields the expected result of nonadditivity. These results are entirely confirmed by a stochastic simulation of the process as summarized in Fig. 5. Notice that when $q_1=q_2$ the process becomes time homogeneous and one recovers the results of Sec. III A.

V. DETERMINISTIC DYNAMICAL SYSTEMS

The principal signature of deterministic dynamics is that the state at any given time is determined uniquely by the knowledge of the state at some other time, chosen as the "initial" time. Deterministic dynamical systems are thus infinite memory systems, sharing in this respect some of the features of systems undergoing non-Markovian processes. Still, while in the latter the required knowledge of the past pertains to a whole set of states, in the former a single state—however remote from (or close to) the present state—is sufficient. Furthermore, contrary to stochastic dy-

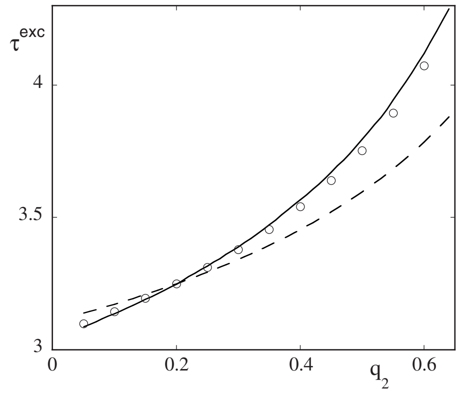

FIG. 5. Mean time between two successive exceedences versus parameter q_2 in the case of a two-state nonhomogeneous Markov process [Eqs. (38), (39), and (41)]. Full line depicts the result obtained by direct stochastic simulation of the process after averaging over 10^5 realizations. Empty circles stand for the analytical expression and dashed line is the sum of the mean recurrence times of the corresponding states. Parameter values are $\beta=0.5$ and $q_1=0.2$.

namics in which the transition probabilities are smooth, in deterministic dynamics the transition between two states associated to two distinct points in phase space has the singular form of a delta function defined along the trajectory joining these states.

In the present section we summarize some results on recurrence and repeated exceedence processes for the class of deterministic systems defined by iterative maps in the interval,

$$x_{n+1} = f(x_n, \mu), \quad a \le x \le b, \quad a \le f(x) \le b, \tag{43}$$

where μ is a control parameter. We consider, successively, mappings giving rise to fully developed chaos and to weak chaos in the form of intermittent behavior. In each case we divide the interval $[a,b]$ into two cells C, $\bar{C}$ such that $C \cup \bar{C} = [a,b]$, playing the role of "subthreshold" and "superthreshold" values of x (see Introduction), and monitor the transitions between C and $\bar{C}$ corresponding to the events defined in Figs. 1 and 2.

A. Fully developed chaos

We consider the logistic map $f(x) = 1-2x^2$, $-1 \le x \le 1$. The partition $C = [-1, x_1]$, $\bar{C} = (x_1, 1)$ is a Markov partition [4] for $x_1 = 0$ and $x_1 = 0.5$ (the fixed point of the mapping) and a non-Markov one for $x_1 = 0.25$. As expected, the probability $W^{\mathrm{exc}}(n,m)$ splits into two independent recurrent events in C and $\bar{C}$ in the first two cases. Still, it is worth mentioning the presence of strong selection rules induced by the deterministic character of the dynamics entailing that for $x_1 = 0.5$ the only possible values of sojourn times in $\bar{C}$ are $n=1$. On the other hand for the non-Markovian partition at $x_1 = 0.25$ statistical independence fails, as one can check explicitly by applying the method of Contingency Tables described in the previous section.

Figure 6 depicts the dependence of the probability of the time u between two successive exceedences as a function of u, for $x_1 = 0$ and $x_1 = 0.25$. In both cases $P(u)$ falls off exponentially, the difference being a faster decay for $x_1 = 0.25$ owing presumably to a more restricted range of allowed values of the exceedence times.

B. Intermittent chaos

We consider the cusp map $f(x) = 1-2|x|^{1/2}$, $-1 \le x \le 1$ and a partition such that the leftmost cell C contains the marginally stable fixed point at $x = -1$ [4], chosen hereafter to correspond to the choice of the boundary $x_1 = 0$. Owing to the weakly chaotic character of the dynamics the probability $W^{\mathrm{exc}}(n,m)$ does not split into two independent recurrent events in C and $\bar{C}$, as checked again explicitly by applying the method of Contingency Tables. The behavior of the probability $P(u)$ of the time between two successive exceedences is depicted in Fig. 7. $P(u)$ falls off now as u^{-2}, in accord with previous results on repeated recurrences by Balakrishnan and the present authors [13]. Here again, the presence of strong selection rules imposed by the deterministic character of the dynamics should be stressed. In particular, the range of allowed sojourn times in cell $\bar{C}$ is much narrower than in cell C.

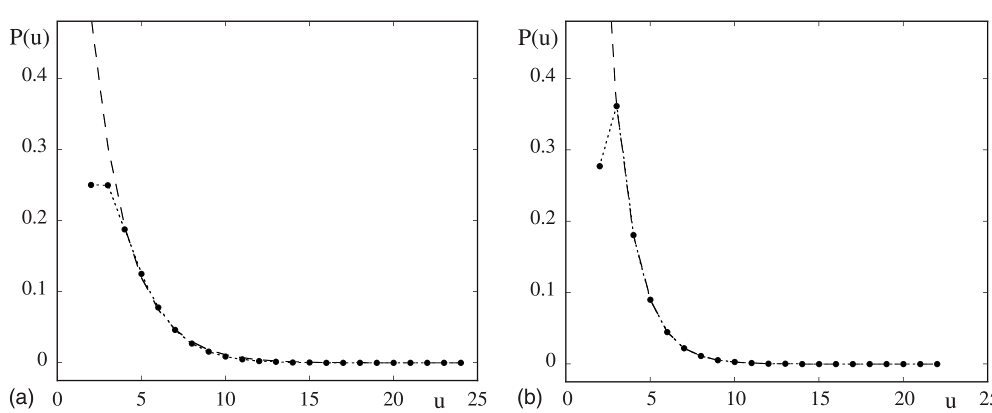

FIG. 6. Dependence of the probability P of the time u between two successive exceedences of the logistic map as a function of u with $x_1 = 0$ (a) and $x_1 = 0.25$ (b) as obtained numerically after averaging over 10^6 realizations. Dashed line represents a best fit with an exponential function.

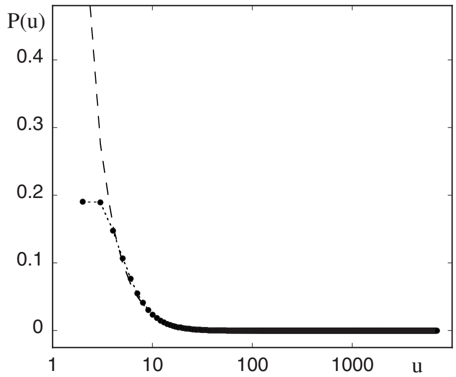

FIG. 7. As in Fig. 6(a) but for the cusp map.

VI. CONCLUSIONS

Recurrent and extreme events are of considerable importance when dealing with the issue of prediction of complex systems. Ordinarily, they are studied by purely statistical tools whereby the set of their successive values is supposed to constitute independent identically distributed random variables (i.i.d.r.v.'s). In this work we have adopted a dynamical approach and analyzed the role of memory in the probabilistic properties of recurrences and of successive exceedences past a threshold. Three types of prototypical systems have been considered: homogeneous first and second order Markov chains, in which memory is limited to a single step or to two steps backward; non-Markovian and nonhomogeneous processes, in which memory extends over the entire past history; and deterministic dynamics, where the knowledge of the initial condition determines uniquely the future states. Our main conclusion has been that there are substantial differences with the i.i.d.r.v. based scenarios. In the Markovian case i.i.d.r.v. like properties such as factorization of the probability of two successive exceedences into recurrence probabilities and the additivity of the corresponding means and variances still hold but, in contrast with the i.i.d.r.v. case, the values of the moments are no longer determined entirely by the invariant measure of the process. In the non-Markovian case factorization fails along the additivity of the variances, but the mean values still satisfy the additivity property. In the nonhomogeneous case nonfactorizability is already manifested through the nonadditivity of the mean values. Finally, in the case of deterministic dynamics one witnesses the presence of strong selection rules limiting the ranges of values of recurrence and successive exceedence times. Factorization and additivity properties may still hold when transitions are between the cells of a Markov partitioning. These conclusions call for a reassessment of the time-honored view, in which analysis and prediction of recurrence and exceedence events rests upon a purely statistical i.i.d.r.v. perspective.

Throughout this work we argued in terms of discrete time processes. Although legitimate at first sight in view of the time-discrete nature of the type of events considered, this approach is nevertheless subjected to a number of limitations. The basic evolution laws governing the observables of a physical system (Newton's or Navier-Stokes equations, etc.) and their probability distributions (Liouville, master or Fokker-Planck equations) are continuous in both time and state space. They do reduce under certain conditions to coarse-grained, space and time-discrete forms upon projection on a submanifold or sheer discretization of time and/or space derivatives [14], but such a reduction usually implies loss of information of some sort. It would be interesting to extend our analysis to this more general setting.

ACKNOWLEDGMENTS

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